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Metoda clasica de comparare a 2 expresii logaritmice

Pas 1. Se incearca ca in fiecare expresie sa apara o singura necunoscuta. Fie acestea a si b

Pas 2. Daca putem define o functie $f(x)$ astfel incat cele 2 expresii sa fie $f(a) si\ f(b)$ atunci stabilim monotonia functiei $f(x)$

Pas 3. Comparam a cu b si avem situatia

$a > b \qquad f(x) strict\ crescatoare\ \Rightarrow f(a) > f(b)$

$\qquad\qquad f(x) strict\ descrescatoare\ \Rightarrow f(a) < f(b)$

Pas 1. $x = \log_5 6 + \log_6 5 = \log_5 6 + \frac{1}{\log_5 6}, necunoscuta\ \log_5 6 = a$

$y = \log_6 7 + \log_7 6 = \log_6 7 + \frac{1}{\log_6 7}, necunoscuta\ \log_6 7 = b$

Pas 2. Definim functia $f(t) = t + \frac{1}{t}$. Avem de comparat $\log_5 6\ cu\ \log_6 7$

$R = \frac{t_2 + \frac{1}{t_2} - t_1 - \frac{1}{t_1}}{t_2 - t_1} = \frac{(t_2 - t_1) - \frac{t_2 - t_1}{t_1 t_2}}{t_2 - t_1} = 1 - \frac{1}{t_1 t_2}$

Cum $\log_5 6\ si\ \log_6 7 > 1 \Rightarrow t_1 t_2 > 1 \Rightarrow R > 0 \Rightarrow f\ strict\ crescatoare$

Pas 3. Sa aratam ca: $\log_5 6 > \log_6 7 \Leftrightarrow \frac{1}{\log_6 5} > \log_6 7 \Leftrightarrow 1 > \log_6 5 \cdot \log_6 7$ (*)

Avem de demonstrat inegalitatea (*). Aplicam inegalitatea mediilor

$\log_6 5 \cdot \log_6 7 < \frac{(\log_6 5 \cdot \log_6 7)^2}{4} = \frac{(\log_6 35)^2}{4} < \frac{(\log_6 36)^2}{4} = \frac{(\log_6 6^2)^2}{4} = 1$

Cum $\log_5 6 > \log_6 7$ si $f(x)$ strict crescatoare $\Rightarrow f(\log_5 6) > f(\log_6 7)$

A. a) Fiind baza logaritmului expresia $1 \cdot 2 \cdot ... \cdot n$, se impune schimbarea bazei. Fiind inegalitate se impune folosirea inegalitatii mediilor. Vom avea:

$\sqrt[n]{1 \cdot 2 \cdot ... \cdot n} \leq \frac{1+2+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$ *Cum $1 \cdot 2 \cdot\cdot n > 1 \Rightarrow \log_{1\cdot 2\cdot ...\cdot n} x$ strict cresc.

$\frac{n+1}{2} > \sqrt[n]{1 \cdot 2 \cdot ... \cdot n} \Rightarrow \log_{1\cdot 2\cdot ...\cdot n} \frac{n+1}{2} > \log_{1\cdot 2\cdot ...\cdot n} \sqrt[n]{1 \cdot 2 \cdot ... \cdot n} = \ = \frac{1}{n} \log_{1\cdot 2\cdot ...\cdot n} 1 \cdot 2 \cdot ... \cdot n = \frac{1}{n}$

b) $\Rightarrow x + \frac{1}{x} \geq 2 \cdot \sqrt[2]{x \frac{1}{x}} = 2 \Rightarrow 2^x + \frac{1}{2^x} \geq 2, cum \ \cos\frac{x}{3} \leq 1 \Rightarrow 2\cos\frac{x}{3} \leq 2 \Rightarrow 2^x + \frac{1}{2^x} = 2 \Rightarrow$

$\Rightarrow (2^x - 1)^2 = 0 \Rightarrow 2^x = 1 \Rightarrow x = 0$

B.a) $f_1 = h_1 o g_1, cu\ h_1 = \sqrt[3]{x} \nearrow, g_1 = x + 3 \nearrow \Rightarrow propr\ 2.a\ f_1 \nearrow$

$f_2 = h_2 o g_2, cu\ h_2 = \sqrt{x} \nearrow, g_2 = x + 2 \nearrow \Rightarrow propr\ 2.a\ f_2 \nearrow$

$\Rightarrow$ propr 3) $f_1 + f_2 = 2$ are o singura solutie $x = 0$

b) $\frac{x^2+1}{x} = x + \frac{1}{x} \geq 2 \cdot \sqrt[2]{x \frac{1}{x}} = 2$ $\quad$ $*h_3 = x(2-x) = -x^2 + 2x$ este concave cu maxim in $x = -\frac{b}{2a} = -\frac{2}{-2} = 1,\ de\ valoare -1 + 2 = 1$; $f_3 = 2^{h_3}$ este strict crescatoare si ia valoarea maxima cand h_3 este maxima, deci pt

$x = 1, h_3 = 1 \Rightarrow f_{3_{max}} = 2^1 = 2 \Rightarrow f_3 \leq 2$ **Din * si ** $\Rightarrow x + \frac{1}{x} = 2 \Rightarrow x = 1\ sing.solutie.$

C. **a**) Este mereu indicat sa lucram in aceiasi baza, de regula 10 sau e

$1 = \log_a a = x \log_a bc \Rightarrow x = \frac{1}{\log_a bc} = \frac{\log a}{\log bc} \Rightarrow \frac{1}{x+1} = \frac{1}{\frac{\log a}{\log bc}+1} = \frac{\log bc}{\log a + \log bc} = \frac{\log b + \log c}{\log a + \log b + \log c}$

$\Rightarrow \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = \frac{2\log a + 2\log b + 2\log c}{\log a + \log b + \log c} = 2$

b)Avem inegalitate, deci ne gandim la inegalitatea mediilor.

Avand $\frac{1}{m} + \frac{1}{n}$ ne gandim la media armonica, mai ales ca avem la monitor $m + n$

$\frac{2}{\frac{1}{m}+\frac{1}{n}} \leq \frac{m+n}{2} \Rightarrow \frac{1}{m} + \frac{1}{n} \geq \frac{4}{m+n}$ deci $\frac{1}{x+1} + \frac{1}{y+1} \geq \frac{4}{x+y+2} \Rightarrow \frac{1}{x+y+2} \leq \frac{1}{4}(\frac{1}{x+1} + \frac{1}{y+1})$

Analog pentru (yz) si (xz). Adunand avem

$\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{x+z+2} \leq \frac{1}{4}\left(\frac{1}{x+1} + \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{z+1}\right) = \frac{1}{4} \cdot 4$ (s-a tinut seama de pct a)

Avem $\log_x y = \frac{\log_z y}{\log_z z} = \log_z y \cdot \log_x z$ deci
$\log_a(\log_a b) = \log_a b \cdot \log_b(\log_a b)$ *
Daca $a < b$; $a, b > 1 \Rightarrow \log_a b > 1 \Rightarrow \log_b(\log_a b) > 0$
Deci ecuatia * este de forma $z = x \cdot y\ cu\ x, y, z > 0 \Rightarrow z > y$
Deci $\log_a(\log_b z) > \log_b(\log_a b)$
Daca $a > b; a, b > 1 \Rightarrow a < \log_a b < 1 \Rightarrow \log_b(\log_a b) < 0$
Deci ecuatia * este de forma $z = x \cdot y, cu\ , y, z < 0\ si\ a < x < 1 \Rightarrow z > y$
$-8 = \frac{1}{2} \cdot (-16)$ deci, $\log_a(\log_b z) > \log_b(\log_a b)$
$h_1(x) = \log_2 x\ \nearrow, g_1(x) = 1 + \sqrt{x}\ \nearrow, m_1(x) = \frac{5^x}{3^x + 4^x}$
$m_1(x) = \frac{1}{\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x}$ unde avem $\left(\frac{3}{5}\right)^x \searrow, \left(\frac{4}{5}\right)^x \searrow, deci\ \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x \searrow \Rightarrow \frac{1}{\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x} \nearrow$
$, m_1(x) \nearrow$
$f_1 = h_1 o g_1 o m_1 = h_1 o (g_1 o m_1) = \nearrow o (\nearrow o \nearrow) = \nearrow o \nearrow = \nearrow \Rightarrow f_1\ strict\ crescatoare$
$h_2(x) = \log_{\frac{1}{2}} x\ \searrow, g_2(x) = 1 + \sqrt{x}\ \nearrow, m_2(x) = \frac{25^x}{24^x + 7^x}$
$m_2(x) = \frac{1}{\left(\frac{24}{25}\right)^x + \left(\frac{7}{25}\right)^x}$ unde avem $\left(\frac{24}{25}\right)^x \searrow, \left(\frac{7}{25}\right)^x \searrow, deci\ \left(\frac{24}{25}\right)^x \searrow, \left(\frac{7}{25}\right)^x \searrow \Rightarrow \frac{1}{\left(\frac{24}{25}\right)^x \searrow, \left(\frac{7}{25}\right)^x} \nearrow,$
$, m_2(x) \nearrow$
$f_2 = h_2 o g_2 o m_2 = h_2 o (g_2 o m_2) = \searrow o (\nearrow o \nearrow) = \searrow o \nearrow = \searrow \Rightarrow f_2\ strict\ descrescatoare$
$\Rightarrow -f_2\ \nearrow$
Ecuatia noastra este $f_1 - f_2 + x = 4\ \Rightarrow \nearrow + \nearrow + \nearrow \equiv \nearrow$ deci ecuatia este o functie strict monotona egala cu o constanta, deci are o singura solutie, $x = 2$
Obs: $\nearrow \equiv strict\ crescatoare$
$\searrow \equiv strict\ descrescatoare$